PHYSICS OF MATERIALS

Physics School Autumn 2025

Series 3 Solutions **26 September 2025**

Exercise 1: Elastic equilibrium condition in a two-phase material

We aim to demonstrate that the volume average stress in a solid at equilibrium, under the condition of zero external forces, is equal to zero.

$$\int_{\Omega} \sigma_{ij}\, dV = 0$$

We have the following conditions:

i) $\sigma_{ik} n_k = 0$ where n_k represents the normal to the surface.

ii) $\frac{\partial \sigma_{ik}}{\partial x_k} = 0$: the solid s equilibrium condition (equation 3.26 of the course)

For the demonstration, we use a small calculation ansatz.

$$\int_{\Omega} \sigma_{ij}\, dV = \int_{\Omega} \sigma_{ik} \delta_{kj}\, dV = \int_{\Omega} \sigma_{ik} \frac{\partial x_j}{\partial x_k} dV \qquad (1)$$

By integrating by parts, the expression (1) becomes:

$$\int_{\Omega} \sigma_{ik} \frac{\partial x_j}{\partial x_k} dV = \int_{\Sigma} x_j \sigma_{ik}\, dS_k - \int_{\Omega} x_j \frac{\partial \sigma_{ik}}{\partial x_k} dV = \int_{\Sigma} x_j \sigma_{ik} n_k\, dS - \int_{\Omega} x_j \frac{\partial \sigma_{ik}}{\partial x_k} dV \qquad (2)$$

The surface integral is zero because of condition i), and the volume integral is also zero because of the local equilibrium condition ii).

For a two-phase composite material containing a volume Ω_1 of phase 1 and a volume Ω_2 of phase 2, we can write:

$$\int_{\Omega} \sigma_{ij}\, dV = 0 = \frac{\Omega_1}{\Omega_1} \int_{\Omega_1} \sigma_{ij}\, dV + \frac{\Omega_2}{\Omega_2} \int_{\Omega_2} \sigma_{ij}\, dV \qquad (3)$$

and thus by posing $\frac{1}{\Omega_1} \int_{\Omega_1} \sigma_{ij}\, dV = \langle \sigma_{ij} \rangle_1$:

$$\Omega_1 \left\langle \sigma_{ij} \right\rangle_1 + \Omega_2 \left\langle \sigma_{ij} \right\rangle_2 = 0 \tag{4}$$

By dividing (4) by Ω and rewriting in terms of the volume fraction (f) of phase 2, we get:

$$(1-f)\left\langle \sigma_{ij} \right\rangle_1 + f\left\langle \sigma_{ij} \right\rangle_2 = 0 \tag{5}$$

This formula can be generalized to a multi-phase material made of n phases:

$$\sum_n f_n \left\langle \sigma_{ij} \right\rangle_n = 0 \tag{6}$$

Exercise 2: Strain tensor in cylindrical and spherical coordinates

The length of a segment after distortion is given by:

$$d\vec{x}' = dx + d\vec{u}$$

Following equation (3.17) of the course notes,

$$dl'^2 - dl^2 \approx 2d\vec{x}d\vec{u} = 2u_{ik}dx_i dx_k$$

we can write the tensor u_{jk} in the new basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$:

$$d\vec{x} = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = dr\vec{e}_r + rd\theta\vec{e}_\theta + dz\vec{e}_z$$

Due to dimension homogeneity, the new variables are:

$$dr,\ rd\theta,\ dz$$

$$\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_z\vec{e}_z$$
$$d\vec{u} = du_r\vec{e}_r + du_\theta\vec{e}_\theta + du_z\vec{e}_z + u_r d\vec{e}_r + u_\theta d\vec{e}_\theta + u_z d\vec{e}_z$$

$$d\vec{e}_r = d\theta\vec{e}_\theta \quad d\vec{e}_\theta = -d\theta\vec{e}_r \quad d\vec{e}_z = 0$$

We differentiate with the new coordinates (like in equation 3.3 of the course):

$$du_r = \frac{\partial u_r}{\partial r}dr + \frac{\partial u_r}{\partial \theta}d\theta + \frac{\partial u_r}{\partial z}dz$$
$$du_\theta = \frac{\partial u_\theta}{\partial r}dr + \frac{\partial u_\theta}{\partial \theta}d\theta + \frac{\partial u_\theta}{\partial z}dz$$
$$du_z = \frac{\partial u_z}{\partial r}dr + \frac{\partial u_z}{\partial \theta}d\theta + \frac{\partial u_z}{\partial z}dz$$

and thus:

$$d\vec{x}d\vec{u} = \frac{\partial u_r}{\partial r}dr^2 + \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right)rdrd\theta + \frac{\partial u_r}{\partial z}drdz +$$
$$+\frac{\partial u_\theta}{\partial r}rdrd\theta + \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right)r^2 d\theta^2 + \frac{\partial u_r}{\partial z}rd\theta dz +$$
$$+\frac{\partial u_z}{\partial r}drdz + \frac{1}{r}\frac{\partial u_z}{\partial \theta}rd\theta dz + \frac{\partial u_z}{\partial z}dz^2$$

$$u_{rr} = \frac{\partial u_r}{\partial r} \quad u_{\theta\theta} = \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \quad u_{zz} = \frac{\partial u_z}{\partial z}$$
$$2u_{r\theta} = \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \quad 2u_{rz} = \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \quad 2u_{\theta z} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

Using analogous derivations for spherical coordinates, the new variables are:

$dr,\ rd\theta,\ r\sin\theta d\phi$

$$u_{rr} = \frac{\partial u_r}{\partial r} \quad u_{\theta\theta} = \left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right) \quad u_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta}{r(tg\theta)}$$
$$2u_{r\theta} = \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r}\right) \quad 2u_{r\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{r} - \frac{u_\varphi}{r} \quad \boldsymbol{2u_{\theta\varphi} = \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \theta} - \frac{u_\varphi}{tg\theta} + \frac{1}{sin\theta}\frac{\partial u_r}{\partial \varphi}\right)}$$